

Unambiguous discrimination between two unknown qudit states

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We consider the unambiguous discrimination between two unknown qudit states in n -dimensional ($n \geq 2$) Hilbert space. By equivalence of unknown pure states to known mixed states and with the Jordan-basis method, we demonstrate that the optimal success probability of the discrimination between two unknown states is independent of the dimension n . We also give a scheme for a physical implementation of the programmable state discriminator that can unambiguously discriminate between two unknown states with optimal probability of success.

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I. INTRODUCTION

As a recent development, the possibility of unambiguous discrimination between unknown quantum states can be potentially useful for many application in quantum communication and quantum computing. A universal device that can unambiguously discriminate between two unknown states has been constructed by Bergou and Hillery [1]. It has three registers, labeled A, B and C, and each register can store a qubit that is in some arbitrary state. In their work, it is assumed that register A is prepared in the state $|\psi_1\rangle$, register C is prepared in the states $|\psi_2\rangle$, and register B is guaranteed to be prepared in either $|\psi_1\rangle$ or $|\psi_2\rangle$. Here, $|\psi_1\rangle$ and $|\psi_2\rangle$ are the states to be distinguished, which are both unknown

$$|\psi_1\rangle = a|0\rangle + b|1\rangle, \quad |\psi_2\rangle = c|0\rangle + d|1\rangle, \quad (1)$$

where all the parameters a, b, c and d are arbitrary unknown complex variables satisfying the normalization equation $|a|^2 + |b|^2 = 1$ and $|c|^2 + |d|^2 = 1$. Furthermore, it is assumed that register B is prepared in the state $|\psi_1\rangle$ with probability η_1 and in the state $|\psi_2\rangle$ with probability η_2 , such that $\eta_1 + \eta_2 = 1$, which guarantees that the state in register B is always one of these two states. The states viewed as a program which are sent into registers A and C (called program registers) are called program states, while the unknown state to be conformed which is sent into the register B (called data register) is called data state. The device constructed here can measure the total input states

$$\begin{aligned} |\Psi_1\rangle &= |\psi_1\rangle_A |\psi_1\rangle_B |\psi_2\rangle_C, \\ |\Psi_2\rangle &= |\psi_1\rangle_A |\psi_2\rangle_B |\psi_2\rangle_C, \end{aligned} \quad (2)$$

which are prepared with apriori probabilities η_1 and η_2 . With the symmetry properties of the input states, this device will then, with some probabilities of success, tell us whether the unknown state in the data register B matches the state stored in program register A or C.

In later works, a series of new devices have been introduced and widely discussed by other authors [2–7].

In these schemes we may have n_A and n_C copies of the states in the program register A and C respectively, and n_B copies of the states in the data register B, then our task is to distinguish the two states

$$\begin{aligned} |\Psi_1\rangle &= |\psi_1\rangle_A^{\otimes n_A} |\psi_1\rangle_B^{\otimes n_B} |\psi_2\rangle_C^{\otimes n_C}, \\ |\Psi_2\rangle &= |\psi_1\rangle_A^{\otimes n_A} |\psi_2\rangle_B^{\otimes n_B} |\psi_2\rangle_C^{\otimes n_C}, \end{aligned} \quad (3)$$

with the minimum error or with the minimum inconclusive probability, if the minimum error strategy or optimum unambiguous strategy is applied, respectively [8].

In both previous papers [2] and [8], it has been proved that the optimal success probability of discrimination between two unknown qubits is an increasing function of n_B , the number of copies in data register B. In this paper, we study the qudit states in n -dimensional Hilbert space and demonstrate that the optimal success probability of discrimination between two unknown states is independent of the dimension n . Unlike the discrimination between two known states, we cannot consider the subspace spanned by the two states only, but we should consider the full n -dimensional space, as the two states are completely unknown to us. We adopt the equivalence between the discrimination of unknown pure states and that of known mixed states, and then reduce the problem to the unambiguous discrimination between two pure states with Jordan-basis method. Finally, we get the optimal success probability of the unambiguous discrimination between the two mixed states and give the detection operators. The detection operators are also applicable to the discrimination between the pure states without averaging over them.

The organization of the paper is as follows. In Sec. II, we give a brief description of the programmable states discrimination, and adopt the equivalence between the discrimination of unknown states to that of known mixed states. In Sec. III, we introduced the Jordan basis for the mean input states. In Sec. IV, we find the optimal unambiguous discrimination and introduce its implementation. Finally, a brief summary is given in Sec. V.

II. THE PROGRAMMABLE DISCRIMINATION

In previous works, states such as $|\psi\rangle = \cos(\phi/2)|0\rangle + \sin(\phi/2)|1\rangle$ have been considered [9]. These states lie in a two-dimensional Hilbert space. Here we introduce a more generalized state

$$|\psi\rangle = \sum_{i=1}^n a_i |i\rangle, \quad (4)$$

where a_i is an arbitrary unknown complex variable. All these states span a n -dimensional Hilbert space \mathcal{H} , for which $|i\rangle$ form a mutually orthogonal basis. The two states we want to distinguish are denoted by $|\psi_1\rangle$ and $|\psi_2\rangle$. As we mentioned before, a copy of each of the two unknown states is provided for the program registers A and C, denoted as $|\psi_1\rangle_A$ and $|\psi_2\rangle_C$, respectively. The state to be confirmed is provided for the data register B as input. We assume that the state in the data register B is guaranteed to be prepared in $|\psi_1\rangle$ with probability η_1 and in $|\psi_2\rangle$ with probability $\eta_2 = 1 - \eta_1$ respectively. Thus we have two possible inputs

$$\begin{aligned} |\Psi_1\rangle &= |\psi_1\rangle_A |\psi_1\rangle_B |\psi_2\rangle_C, \\ |\Psi_2\rangle &= |\psi_1\rangle_A |\psi_2\rangle_B |\psi_2\rangle_C. \end{aligned} \quad (5)$$

Here, $|\psi_1\rangle$ and $|\psi_2\rangle$ are completely arbitrary states

$$\begin{aligned} |\psi_1\rangle &= \sum_{i=1}^n a_i |i\rangle, \\ |\psi_2\rangle &= \sum_{i=1}^n b_i |i\rangle, \end{aligned} \quad (6)$$

that remain unknown throughout the entire discrimination process.

Since the inputs $|\Psi_1\rangle$ and $|\Psi_2\rangle$ are unknown, and the states $|\psi_1\rangle$ and $|\psi_2\rangle$ making up these two inputs can change from preparation to preparation, it is only the pattern, middle states (B) matching the first (A) or the last (C) state that is preserved from one preparation to the next. Therefore, we can introduce the corresponding density operators

$$\begin{aligned} \rho_1 &= \{|\Psi_1\rangle\langle\Psi_1|\}_{av}, \\ \rho_2 &= \{|\Psi_2\rangle\langle\Psi_2|\}_{av}, \end{aligned} \quad (7)$$

where the average is taken over the entire parameter space of states $|\psi_1\rangle$ and $|\psi_2\rangle$. Here, we should notice that the optimal strategy for discrimination between the two states is the strategy that is optimal on average. That's to say, we can unambiguously discriminate between $|\psi_1\rangle$ and $|\psi_2\rangle$ as soon as we can unambiguously discriminate between ρ_1 and ρ_2 .

To this end, we now define the space and operators that we will need. Let \mathcal{H} be the n -dimensional space for the unknown states and then the full space is $\otimes^3 \mathcal{H}$. The space of symmetric states in $\mathcal{H} \otimes \mathcal{H}$ is denoted by Σ , which

is an $n(n+1)/2$ -dimensional subspace. $|\Psi_1\rangle$ is an element of $S_1 = \Sigma \otimes \mathcal{H}$ and $|\Psi_2\rangle$ is an element of $S_2 = \mathcal{H} \otimes \Sigma$. Therefore, $S_0 = S_1 \cap S_2$, the intersection of S_1 and S_2 , is the space of symmetric states in the full space $\otimes^3 \mathcal{H}$. S_0 is a subspace of dimension $n(n+1)(n+2)/6$, while S_1 and S_2 both $n^2(n+1)/2$. Let S_3 be the subspace of $\otimes^3 \mathcal{H}$ generated by S_1 and S_2 , and the dimension of S_3 is $n(n+1)(5n-2)/6$. Let S_4 be the orthogonal complement of S_0 in S_1 , let S_5 be the orthogonal complement of S_0 in S_2 , and let S_6 be the orthogonal complement of S_0 in S_3 .

Clearly, the average in ρ_1 uniformly fills the symmetric subspace of A and B and the entire subspace of C, whereas the average in ρ_2 uniformly fills the symmetric subspace of B and C and the entire subspace of A. Therefore, the corresponding density operators averaged over the unknown states, can be expressed as

$$\begin{aligned} \rho_1 &= \frac{2}{n(n+1)} P_\Sigma \otimes \frac{1}{n} I_C \\ &= \frac{2}{n^2(n+1)} P_\Sigma \otimes I_C, \\ \rho_2 &= \frac{1}{n} I_A \otimes \frac{2}{n(n+1)} P_\Sigma \\ &= \frac{2}{n^2(n+1)} I_A \otimes P_\Sigma, \end{aligned} \quad (8)$$

where P_Σ is the projection onto Σ , and I onto \mathcal{H} . P_Σ can be expressed as

$$P_\Sigma = \sum_{i \leq j=1}^n |u_{ij}^{(2)}\rangle \langle u_{ij}^{(2)}|, \quad (9)$$

where $|u_{ij}^{(2)}\rangle$ ($i \leq j = 1, \dots, n$) are the unique unit vectors in the symmetric subspace Σ ,

$$\begin{aligned} |u_{11}^{(2)}\rangle &= |11\rangle, \\ |u_{12}^{(2)}\rangle &= \frac{1}{\sqrt{2}}(|12\rangle + |21\rangle), \\ &\vdots \\ |u_{nn}^{(2)}\rangle &= |nn\rangle. \end{aligned} \quad (10)$$

Consequently, we reduced the problem to discriminating between the $[n^2(n+1)/2]$ -dimensional spaces S_1 and S_2 in S_3 , which is equivalent to discriminating between subspaces S_4 and S_5 in the $[2n(n+1)(n-1)/3]$ -dimensional space S_6 .

III. JORDAN BASIS

It has been shown in [10] that two nonorthogonal subspaces \mathcal{H}_1 and \mathcal{H}_2 of a Hilbert space can be unambiguously discriminated if we can find their canonical or Jordan bases [11]. The definition of the Jordan bases is as

follows. The set of orthogonal and normalized basis vectors $\{|g_1\rangle, \dots, |g_n\rangle\}$ in \mathcal{H}_1 and $\{|h_1\rangle, \dots, |h_n\rangle\}$ in \mathcal{H}_2 form Jordan bases if and only if

$$\langle g_i | h_j \rangle = \delta_{ij} \cos \theta_i, \quad (11)$$

where θ_i are the Jordan angles ($\theta_1 \leq \theta_2 \leq \dots \leq \theta_n$). In the Jordan form, the full Hilbert space is decomposed into n orthogonal subspaces. Here, we denote the i th subspace spanned by $|g_i\rangle$ and $|h_i\rangle$ by T_i . Clearly, if the dimension of T_i is 1, we can get $\cos \theta_i = 1$, and no further discrimination is possible in T_i . On the other hand, if $\cos \theta_i < 1$, $|g_i\rangle$ and $|h_i\rangle$ are linearly independent,

and they can be distinguished as two known pure states, which is familiar to us. Thus our task is reduced to some separate discriminations in each subspace, between two known pure states.

Now, we will choose some bases to construct Jordan bases for the density operators ρ_1 and ρ_2 . First, we should notice that the fully symmetric subspace of the three states S_0 must be common to both inputs. Here, we denote the basis for the subspace S_0 by $|u_{ijk}^{(3)}\rangle$, where $1 \leq i \leq j \leq k \leq n$ is satisfied, and there are $n(n+1)(n+2)/6$ these unique unit vectors, which can be expressed as follows,

$$|u_{ijk}^{(3)}\rangle = \begin{cases} |ijk\rangle & \text{if } i = j = k \\ \frac{|ijk\rangle + |ikj\rangle + |kij\rangle}{\sqrt{3}} & \text{if } i = j \neq k \\ \frac{|ijk\rangle + |jik\rangle + |jki\rangle}{\sqrt{3}} & \text{if } i \neq j = k \\ \frac{|ijk\rangle + |jik\rangle + |ikj\rangle + |kij\rangle + |jki\rangle + |kji\rangle}{\sqrt{6}} & \text{if } i \neq j \neq k \end{cases}. \quad (12)$$

Then, the structure of the two density operators in Eq. (8), in particular the decomposition on the right-hand side, suggests that we consider $|u_{ij}^{(2)}\rangle \otimes |\alpha\rangle$ and $|\alpha\rangle \otimes |u_{ij}^{(2)}\rangle$, where $\alpha = 1, \dots, n$ and $1 \leq i \leq j \leq n$. These vectors form orthogonal bases for S_1 and S_2 , respectively. Thus the

unique unit vector $|u_{ijk}^{(3)}\rangle$ in S_0 can be expressed in terms of either S_1 or S_2 , since it is in both spaces. Here, we neglect the vectors $|u_{ijk}^{(3)}\rangle$ that satisfy $i = j = k$, and a direct calculation shows that

$$|u_{ijk}^{(3)}\rangle = \begin{cases} \sqrt{\frac{2}{3}}|u_{ik}^{(2)}\rangle|j\rangle + \sqrt{\frac{1}{3}}|ijk\rangle & \text{if } i = j \neq k \\ \sqrt{\frac{2}{3}}|u_{ij}^{(2)}\rangle|k\rangle + \sqrt{\frac{1}{3}}|jki\rangle & \text{if } i \neq j = k \\ \frac{1}{\sqrt{3}}(|u_{ij}^{(2)}\rangle|k\rangle + |u_{ik}^{(2)}\rangle|j\rangle + |u_{jk}^{(2)}\rangle|i\rangle) & \text{if } i \neq j \neq k \end{cases}, \quad (13)$$

and

$$|u_{ijk}^{(3)}\rangle = \begin{cases} \sqrt{\frac{2}{3}}|i\rangle|u_{jk}^{(2)}\rangle + \sqrt{\frac{1}{3}}|kij\rangle & \text{if } i = j \neq k \\ \sqrt{\frac{2}{3}}|j\rangle|u_{ik}^{(2)}\rangle + \sqrt{\frac{1}{3}}|ijk\rangle & \text{if } i \neq j = k \\ \frac{1}{\sqrt{3}}(|k\rangle|u_{ij}^{(2)}\rangle + |j\rangle|u_{ik}^{(2)}\rangle + |i\rangle|u_{jk}^{(2)}\rangle) & \text{if } i \neq j \neq k \end{cases}. \quad (14)$$

We can now introduce the vectors

$$|g_{ijk}\rangle = \sqrt{\frac{1}{3}}|u_{ik}^{(2)}\rangle|j\rangle - \sqrt{\frac{2}{3}}|ijk\rangle \quad (15)$$

$$|h_{ijk}\rangle = \sqrt{\frac{1}{3}}|i\rangle|u_{jk}^{(2)}\rangle - \sqrt{\frac{2}{3}}|kij\rangle \quad (16)$$

for $i = j \neq k$,

$$|g_{ijk}\rangle = \sqrt{\frac{1}{3}}|u_{ij}^{(2)}\rangle|k\rangle - \sqrt{\frac{2}{3}}|jki\rangle \quad (17)$$

$$|h_{ijk}\rangle = \sqrt{\frac{1}{3}}|j\rangle|u_{ik}^{(2)}\rangle - \sqrt{\frac{2}{3}}|ijk\rangle \quad (18)$$

for $i \neq j = k$,

$$\begin{aligned} |g_{ijk}\rangle &= \frac{3-\sqrt{3}}{6}|u_{ij}^{(2)}\rangle|k\rangle - \frac{3+\sqrt{3}}{6}|u_{ik}^{(2)}\rangle|j\rangle + \frac{\sqrt{3}}{3}|u_{jk}^{(2)}\rangle|i\rangle \\ |g'_{ijk}\rangle &= \frac{3-\sqrt{3}}{6}|u_{ik}^{(2)}\rangle|j\rangle - \frac{3+\sqrt{3}}{6}|u_{ij}^{(2)}\rangle|k\rangle + \frac{\sqrt{3}}{3}|u_{jk}^{(2)}\rangle|i\rangle \end{aligned} \quad (19)$$

and

$$\begin{aligned} |h_{ijk}\rangle &= \frac{3-\sqrt{3}}{6}|k\rangle|u_{ij}^{(2)}\rangle - \frac{3+\sqrt{3}}{6}|j\rangle|u_{ik}^{(2)}\rangle + \frac{\sqrt{3}}{3}|i\rangle|u_{jk}^{(2)}\rangle \\ |h'_{ijk}\rangle &= \frac{3-\sqrt{3}}{6}|j\rangle|u_{ik}^{(2)}\rangle - \frac{3+\sqrt{3}}{6}|k\rangle|u_{ij}^{(2)}\rangle + \frac{\sqrt{3}}{3}|i\rangle|u_{jk}^{(2)}\rangle \end{aligned} \quad (20)$$

for $i \neq j \neq k$. It is easy to see that $|g\rangle$'s form orthogonal bases for subspace S_4 , while $|h\rangle$'s form orthogonal bases for subspace S_5 . And there are $n(n+1)(n-1)/3$ $|g\rangle$'s and $n(n+1)(n-1)/3$ $|h\rangle$'s. Thus we can rearrange the footnotes, and get $|g_i\rangle$ and $|h_i\rangle$, where i can change from 1 to $n(n+1)(n-1)/3$. From the explicit expressions of $|g_i\rangle$ and $|h_i\rangle$, we can easily get

$$\langle g_i | h_j \rangle = -\frac{1}{2}\delta_{ij}, \quad (21)$$

and $\{|g_i\rangle\}$ and $\{|h_i\rangle\}$ form Jordan bases for S_4 and S_5 . The two density operators that we want to distinguish can now be expressed as

$$\begin{aligned} \rho_1 &= \frac{2}{n^2(n+1)} \left(P_{S_0} + \sum_{i=1}^{i_0} |g_i\rangle\langle g_i| \right), \\ \rho_2 &= \frac{2}{n^2(n+1)} \left(P_{S_0} + \sum_{i=1}^{i_0} |h_i\rangle\langle h_i| \right), \end{aligned} \quad (22)$$

where $i_0 = n(n+1)(n-1)$. P_{S_0} is the projection onto the subspace S_0 , and

$$P_{S_0} = \sum_{i \leq j \leq k=1}^{i_0} |u_{ijk}^{(2)}\rangle\langle u_{ijk}^{(2)}|. \quad (23)$$

Now, let T_i be the two-dimensional space spanned by the nonorthogonal but linearly independent vectors $|g_i\rangle$ and $|h_i\rangle$. The T_i form a decomposition of subspace S_6 into $n(n+1)(n-1)$ mutually perpendicular two-dimensional subspaces. Our next problem is how to distinguish the two pure states in every subspace T_i .

IV. DERIVATION AND IMPLEMENTATION OF THE OPTIMAL UNAMBIGUOUS DISCRIMINATION

We now want to unambiguously discriminate between the subspaces S_4 and S_5 in S_6 . First, we distinguish the

Jordan Basis states $|g_i\rangle$ and $|h_i\rangle$ within their subspace T_i . In subspace T_i , the apriori probabilities of $|g_i\rangle$ and $|h_i\rangle$ are η_1 and η_2 respectively. Here we will use the method mentioned in [18]. Let us define $|g_i^\perp\rangle$ and $|h_i^\perp\rangle$ the reciprocal states of $|g_i\rangle$ and $|h_i\rangle$ respectively, where

$$\langle g_i^\perp | h_i \rangle = \langle h_i^\perp | g_i \rangle = 0 \quad (24)$$

and then

$$\begin{aligned} |g_i^\perp\rangle &= \frac{|g_i\rangle - \langle h_i | g_i \rangle |h_i\rangle}{\sqrt{1 - |\langle h_i | g_i \rangle|^2}} \\ &= \frac{2}{\sqrt{3}}|g_i\rangle + \frac{1}{\sqrt{3}}|h_i\rangle, \\ |h_i^\perp\rangle &= \frac{|h_i\rangle - \langle g_i | h_i \rangle |g_i\rangle}{\sqrt{1 - |\langle g_i | h_i \rangle|^2}} \\ &= \frac{2}{\sqrt{3}}|h_i\rangle + \frac{1}{\sqrt{3}}|g_i\rangle. \end{aligned} \quad (25)$$

Thus, $|g_i\rangle$ and $|h_i\rangle$ can be rewritten as

$$\begin{aligned} |g_i\rangle &= \frac{\sqrt{3}}{2}|g_i^\perp\rangle - \frac{1}{2}|h_i\rangle, \\ |h_i\rangle &= |h_i^\perp\rangle. \end{aligned} \quad (26)$$

We can give a physical implementation based on Neumark's theorem [12]. Following the proposals given in [13–16]: (a) any pure state can be realized by a single-photon state and (b) following Reck's theorem [16], any unitary transformation can also be realized by an optical network consisting of beam-splitters, phase-shifter, etc. [17], we can construct an optical device, which is presented in Fig. 1a, to unambiguously discriminate between $|g_i\rangle$ and $|h_i\rangle$. An additional port is initially prepared in the vacuum state $|0\rangle$. The operation of such a device is described by a unitary matrix U_3 which gives the probability amplitudes for a single photon entering via inputs $|g_i^\perp\rangle$ and $|h_i\rangle$ to leave the device by outputs $|D_1^{(i)}\rangle$, $|D_2^{(i)}\rangle$ and $|F^{(i)}\rangle$. Here, the four-port optical interferometer, which is presented in Fig. 1b, is used and

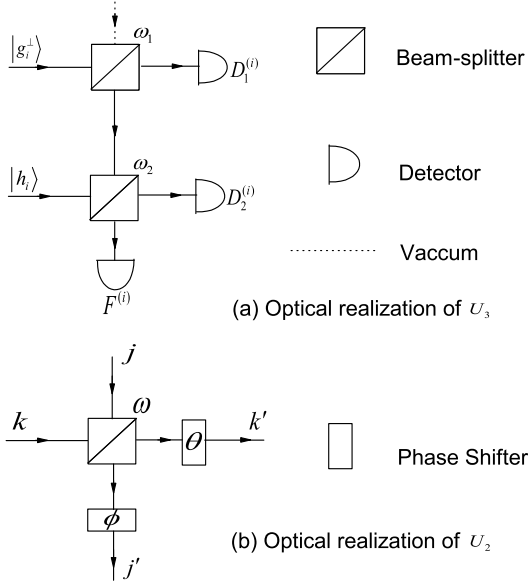


FIG. 1: (a) An optical six-port interferometer used to unambiguously discriminate between $|g_i\rangle$ and $|h_i\rangle$. (b) An optical four-port interferometer which is capable of realizing the two-dimensional unitary transformation defined by Eq. (27).

it is capable of realizing any 2×2 unitary transformation U_2

$$U_2 = \begin{pmatrix} \sin \omega e^{i\phi} & \cos \omega e^{i\phi} \\ \cos \omega e^{i\theta} & -\sin \omega e^{i\theta} \end{pmatrix}, \quad (27)$$

where we have tacitly assumed that the action of all the 2×2 beam splitter upon the states of the photon is described by a unitary matrix of real coefficient

$$\begin{pmatrix} \sin \omega & \cos \omega \\ \cos \omega & -\sin \omega \end{pmatrix}. \quad (28)$$

The parameter ω describes the transmittance ($\sqrt{t} = \sin \omega$) and the reflectivity ($\sqrt{r} = \cos \omega$) of the beam splitter. As shown in Fig. 1a, the action of this device, denoted by U_3 gives,

$$\begin{aligned} U_3|g_i^\perp\rangle &= -\sin \omega_1|D_1^{(i)}\rangle + \cos \omega_1 \cos \omega_2|D_2^{(i)}\rangle \\ &\quad + \cos \omega_1 \sin \omega_2|F^{(i)}\rangle \\ U_3|h_i\rangle &= -\sin \omega_2|D_2^{(i)}\rangle + \cos \omega_2|F^{(i)}\rangle \end{aligned} \quad (29)$$

From Eq. (26), U_3 will transform $|g_i\rangle$ and $|h_i\rangle$ into

$$\begin{aligned} |g_i^{out}\rangle &= U_3|g_i\rangle \\ &= -\frac{\sqrt{3}}{2}\sin \omega_1|D_1^{(i)}\rangle \\ &\quad + \left(\frac{\sqrt{3}}{2}\cos \omega_1 \cos \omega_2 + \frac{1}{2}\sin \omega_2\right)|D_2^{(i)}\rangle \\ &\quad + \left(\frac{\sqrt{3}}{2}\cos \omega_1 \sin \omega_2 - \frac{1}{2}\cos \omega_2\right)|F^{(i)}\rangle \end{aligned} \quad (30)$$

$$\begin{aligned} |h_i^{out}\rangle &= U_3|h_i\rangle \\ &= -\sin \omega_2|D_2^{(i)}\rangle + \cos \omega_2|F^{(i)}\rangle \end{aligned} \quad (31)$$

As shown in Fig. 1a, both $|g_i\rangle$ and $|h_i\rangle$ have their own detectors $D_1^{(i)}$ and $D_2^{(i)}$, and these two detectors will tell us which the input is, while $F^{(i)}$ corresponds to failure. This suggests that no photon can be detected by detector $D_2^{(i)}$ ($D_1^{(i)}$) when the input is $|g_i\rangle$ ($|h_i\rangle$). Thus, from Eq. (30) we get

$$\begin{aligned} \cos^2 \omega_2 &= \frac{1}{1 + 3 \cos^2 \omega_1}, \\ \sin^2 \omega_2 &= \frac{3 \cos^2 \omega_1}{1 + 3 \cos^2 \omega_1}. \end{aligned} \quad (32)$$

Eq. (30) and Eq. (31) are reduced to

$$\begin{aligned} |g_i^{out}\rangle &= U_2|g_i\rangle \\ &= -\frac{\sqrt{3}}{2}\sin \omega_1|D_1^{(i)}\rangle \\ &\quad + \left(\frac{\sqrt{3}}{2}\cos \omega_1 \sin \omega_2 - \frac{1}{2}\cos \omega_2\right)|F^{(i)}\rangle, \quad (33) \\ |h_i^{out}\rangle &= U_3|h_i\rangle \\ &= -\sqrt{\frac{3 \cos^2 \omega_1}{1 + 3 \cos^2 \omega_1}}|D_2^{(i)}\rangle \\ &\quad + \sqrt{\frac{1}{1 + 3 \cos^2 \omega_1}}|F^{(i)}\rangle. \end{aligned} \quad (34)$$

In other words, we can choose

$$\begin{aligned} \Pi_1^{(i)out} &= |D_1^{(i)}\rangle\langle D_1^{(i)}|, \\ \Pi_2^{(i)out} &= |D_2^{(i)}\rangle\langle D_2^{(i)}|, \\ \Pi_0^{(i)out} &= |F^{(i)}\rangle\langle F^{(i)}|, \end{aligned} \quad (35)$$

with

$$\begin{aligned} \Pi_1^{(i)out}|h_i^{out}\rangle &= \Pi_2^{(i)out}|g_i^{out}\rangle = 0, \\ \Pi_1^{(i)out} + \Pi_2^{(i)out} + \Pi_0^{(i)out} &= \mathbf{I}, \end{aligned} \quad (36)$$

to be our detection operators. Here, also from Fig. 1a, we can get

$$\begin{aligned} |D_1^{(i)}\rangle &= -\sin \omega_1|g_i^\perp\rangle, \\ |D_2^{(i)}\rangle &= -\frac{2}{\sqrt{3}}\sin \omega_2|h_i^\perp\rangle, \end{aligned} \quad (37)$$

where Eq. (25) has been used.

By projecting these detection operators back onto the space T_i , we can get

$$\begin{aligned}\Pi_1^{(i)} &= \sin^2 \omega_1 |g_i^\perp\rangle \langle g_i^\perp|, \\ \Pi_2^{(i)} &= \frac{4}{3} \sin^2 \omega_2 |h_i^\perp\rangle \langle h_i^\perp| \\ &= \frac{4 \cos^2 \omega_1}{1 + 3 \cos^2 \omega_1} |h_i^\perp\rangle \langle h_i^\perp|, \\ \Pi_0^{(i)} &= \mathbf{I} - \Pi_1^{(i)} - \Pi_2^{(i)}\end{aligned}\quad (38)$$

which form the POVM detection operators for the unambiguous discrimination between $|g_i\rangle$ and $|h_i\rangle$.

So we find

$$\begin{aligned}\langle g_i | \Pi_1^{(i)} | g_i \rangle &= \sin^2 \omega_1 |\langle g_i | g_i^\perp \rangle|^2 \\ &= \frac{3}{4} \sin^2 \omega_1, \\ \langle h_i | \Pi_2^{(i)} | h_i \rangle &= \frac{4 \cos^2 \omega_1}{1 + 3 \cos^2 \omega_1} |\langle h_i | h_i^\perp \rangle|^2 \\ &= \frac{3 \cos^2 \omega_1}{1 + 3 \cos^2 \omega_1}.\end{aligned}\quad (39)$$

And the probability of successfully identifying the two states is

$$\begin{aligned}P^i(\omega_1) &= \eta_1 \langle g_i | \Pi_1^{(i)} | g_i \rangle + \eta_2 \langle h_i | \Pi_2^{(i)} | h_i \rangle \\ &= \frac{3}{4} \eta_1 \sin^2 \omega_1 + \frac{3 \eta_2 \cos^2 \omega_1}{1 + 3 \cos^2 \omega_1}.\end{aligned}\quad (40)$$

By letting $x = 1 + 3 \cos^2 \omega_1 (1 \leq x \leq 4)$, $P^i(\omega_1)$ can be rewritten as

$$P^i(x) = 1 - \frac{\eta_1}{4}x - \frac{\eta_2}{x}.\quad (41)$$

This function, $P^i(x)$, has the property that $dP^i(x)/dx = 0$ happens at

$$x = x_0 = 2\sqrt{\frac{\eta_2}{\eta_1}}.\quad (42)$$

The optimal value of $P^i(x)$ denoted by $P^{i,opt}$ in the domain $1 \leq x \leq 4$, can be gotten in following three cases:
(a) if $\frac{1}{5} \leq \eta_1 \leq \frac{4}{5}$, the requirement $1 \leq x \leq 4$ is satisfied, thus we have

$$P^{i,opt} = P^i(x = x_0) = 1 - 2\sqrt{\eta_1 \eta_2},\quad (43)$$

(b) if $\eta_1 < \frac{1}{5}$, we should choose

$$P^{i,opt} = P^i(x = 4) = \frac{3}{4} \eta_2,\quad (44)$$

and (c) if $\eta_1 > \frac{4}{5}$,

$$P^{i,opt} = P^i(x = 1) = \frac{3}{4} \eta_1.\quad (45)$$

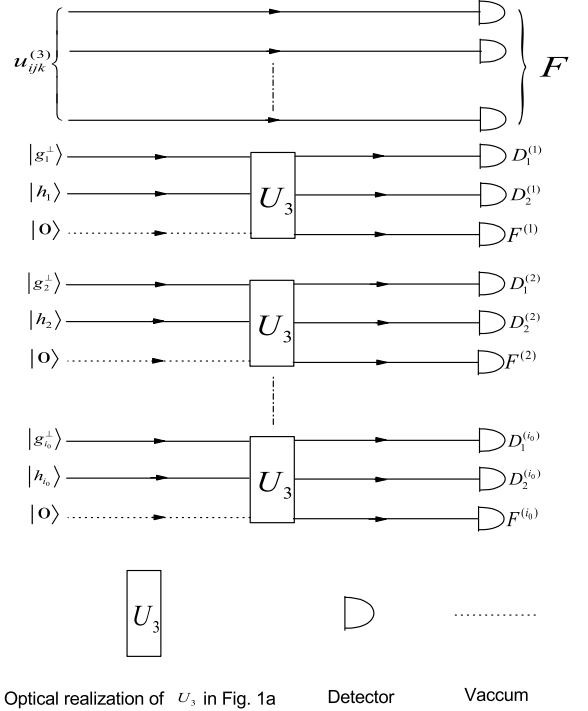


FIG. 2: Implementation of the programmable discriminator between ρ_1 and ρ_2 ($|\Psi_1\rangle$ and $|\Psi_2\rangle$), which consists of each implementation of discrimination between $|g_i\rangle$ and $|h_i\rangle$.

Our finding can be summarized as follows

$$P^{i,opt} = \begin{cases} \frac{3}{4} \eta_2 & \text{if } \eta_1 < \frac{1}{5} \\ 1 - 2\sqrt{\eta_1 \eta_2} & \text{if } \frac{1}{5} \leq \eta_1 \leq \frac{4}{5} \\ \frac{3}{4} \eta_1 & \text{if } \eta_1 > \frac{4}{5} \end{cases}.\quad (46)$$

Now, the discrimination between the two density operators ρ_1 and ρ_2 is to achieve $n(n+1)(n-1)/3$ discriminations described above simultaneously. But, here the probability for the occurrence of the input state in T_i is $p(T_i) = \frac{2}{n^2(n+1)}$. We can also give its implementation in Fig. 2. We leave $|u_{ijk}^{(2)}\rangle$ alone, since they are in both subspaces S_1 and S_2 , and can't be distinguished. The POVM which unambiguously distinguishes S_1 and

S_2 has the form

$$\begin{aligned}
\Pi_1 &= \sum_{i=1}^{i_0} \Pi_1^{(i)} \\
&= \sin^2 \omega_1 \sum_{i=1}^{i_0} |g_i^\perp\rangle \langle g_i^\perp|, \\
\Pi_2 &= \sum_{i=1}^{i_0} \Pi_2^{(i)} \\
&= \frac{4 \cos^2 \omega_1}{1 + 3 \cos^2 \omega_1} \sum_{i=1}^{i_0} |h_i^\perp\rangle \langle h_i^\perp|, \\
\Pi_0 &= \mathbf{I} - \Pi_1 - \Pi_2,
\end{aligned} \tag{47}$$

where $i_0 = n(n+1)(n-1)/3$. Thus, the probability of successfully identifying the two density operators ρ_1 and ρ_2 is

$$\begin{aligned}
\tilde{P}(\omega_1) &= \eta_1 \text{Tr}(\Pi_1 \rho_1) + \eta_2 \text{Tr}(\Pi_2 \rho_2) \\
&= \frac{2 \sin^2 \omega_1}{n^2(n+1)} \sum_{i=1}^{i_0} |\langle g_i | g_i^\perp \rangle|^2 \\
&\quad + \frac{8 \cos^2 \omega_1}{n^2(n+1)(1+3 \cos^2 \omega_1)} \sum_{i=1}^{i_0} |\langle h_i | h_i^\perp \rangle|^2 \\
&= \frac{(n-1)\eta_1}{2n} \sin^2 \omega_1 + \frac{2(n-1)\eta_2 \cos^2 \omega_1}{n(1+3 \cos^2 \omega_1)}.
\end{aligned} \tag{48}$$

Then, the optimal success probability \tilde{P}^{opt} can be expressed as

$$\tilde{P}^{opt} = \begin{cases} \frac{n-1}{2n} \eta_2 & \text{if } \eta_1 < \frac{1}{5} \\ \frac{2(n-1)}{3n} (1 - \sqrt{\eta_1 \eta_2}) & \text{if } \frac{1}{5} \leq \eta_1 \leq \frac{4}{5} \\ \frac{n-1}{2n} \eta_1 & \text{if } \eta_1 > \frac{4}{5} \end{cases}. \tag{49}$$

Now, we come back to the original problem that discrimination between two pure states $|\psi_1\rangle$ ($|\Psi_1\rangle$) and $|\psi_2\rangle$ ($|\Psi_2\rangle$). As mentioned before, the POVM formed by Π_1 , Π_2 and Π_0 with

$$\Pi_1 |\Psi_2\rangle = \Pi_2 |\Psi_1\rangle = 0 \tag{50}$$

can be used. Its implementation is the same as discrimination between ρ_1 and ρ_2 in Fig. 2. The success probability is

$$\begin{aligned}
P(\omega_1) &= \eta_1 \langle \Psi_1 | \Pi_1 | \Psi_1 \rangle + \eta_2 \langle \Psi_2 | \Pi_2 | \Psi_2 \rangle \\
&= \left(\frac{1}{2} \eta_1 \sin^2 \omega_1 + \frac{2 \eta_2 \cos^2 \omega_1}{1 + 3 \cos^2 \omega_1} \right) (1 - |\langle \psi_1 | \psi_2 \rangle|^2),
\end{aligned} \tag{51}$$

where the relationship

$$\begin{aligned}
\sum_{i=1}^{i_0} \langle \Psi_1 | g_i^\perp \rangle \langle g_i^\perp | \Psi_1 \rangle &= \sum_{i=1}^{i_0} \langle \Psi_2 | h_i^\perp \rangle \langle h_i^\perp | \Psi_2 \rangle \\
&= \frac{1}{2} (1 - |\langle \psi_1 | \psi_2 \rangle|^2)
\end{aligned} \tag{52}$$

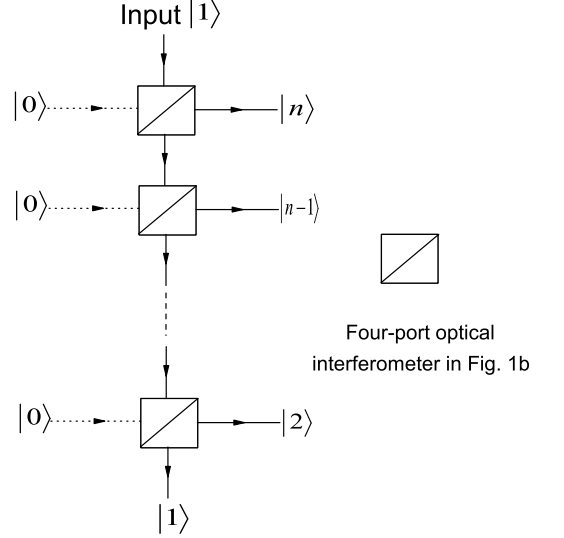


FIG. 3: An optical device to prepare the state $|\psi\rangle = \sum_{i=1}^n |i\rangle$.

has been used. Finally, we can also get the optimal success probability as follows

$$P^{opt} = \begin{cases} \frac{1}{2} \eta_2 (1 - |\langle \psi_1 | \psi_2 \rangle|^2) & \text{if } \eta_1 < \frac{1}{5} \\ \frac{2}{3} (1 - \sqrt{\eta_1 \eta_2}) (1 - |\langle \psi_1 | \psi_2 \rangle|^2) & \text{if } \frac{1}{5} \leq \eta_1 \leq \frac{4}{5} \\ \frac{1}{2} \eta_1 (1 - |\langle \psi_1 | \psi_2 \rangle|^2) & \text{if } \eta_1 > \frac{4}{5} \end{cases}, \tag{53}$$

which is apparently independent of n , the dimension of space \mathcal{H} , as we mentioned at the beginning.

V. DISCUSSION AND CONCLUSION

How to prepare the states with single photon is another important question in the optical realization of the discriminator. We shall give a brief discussion about this. An optical setting shown in Fig. 3 can prepare the state $|\psi\rangle = \sum_{i=1}^n a_i |i\rangle$. It is constructed by a series of 2×2 unitary transformations which can be realized by the four-port optical interferometer in Fig. 1b. If the parameters of each four-port interferometer are fixed properly, this setting achieves a single-photon state in n -dimensional space \mathcal{H} . We denote the operation of this device by $U_{|\psi\rangle}$, and

$$|\psi\rangle = U_{|\psi\rangle} |i\rangle. \tag{54}$$

Then, by using $U_{|\psi_1\rangle}$, $U_{|\psi_?\rangle}$ and $U_{|\psi_2\rangle}$ in succession, we can get

$$\begin{aligned} |\Psi_?\rangle &= |\psi_1\rangle \otimes |\psi_?\rangle \otimes |\psi_2\rangle \\ &= U_{|\psi_2\rangle} U_{|\psi_?\rangle} U_{|\psi_1\rangle} |1\rangle \\ &= U_{|\psi_1\rangle} \otimes U_{|\psi_?\rangle} \otimes U_{|\psi_2\rangle} \cdot |1\rangle \otimes |1\rangle \otimes |1\rangle \quad (55) \end{aligned}$$

where $|\psi_?\rangle$ is either $|\psi_1\rangle$ or $|\psi_2\rangle$ and $|1\rangle \otimes |1\rangle \otimes |1\rangle$ denote the only input for the device. And $|\Psi_1\rangle$ or $|\Psi_2\rangle$ is to be prepared depending on $U_{|\psi_?\rangle}$ being $U_{|\psi_1\rangle}$ or $U_{|\psi_2\rangle}$.

In conclusion, we have reconsidered the problem of the universal programmable quantum state discriminator originally introduced in [1]. And we solved a more generalized problem that the unknown states are qudit states in the n -dimensional ($n \geq 2$) Hilbert space \mathcal{H} .

We adopted the equivalence between the discrimination of unknown pure quantum states and that of known mixed states. With the Jordan-basis method, we simplified the problem and finally achieved the optimal unambiguous discrimination between the two unknown states, and the corresponding detection operators have already been given. Significantly, we arrived at an important conclusion that the optimal success probability of the discrimination between two unknown pure states is independent of n , the dimension of the space \mathcal{H} .

Furthermore, we also give the implementation of the optimal POVM based on the Neumark's theorem. The POVM can be implemented on a larger Hilbert space, where the additional degrees of freedom called ancilla are needed.

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